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Complexity of the Delaunay triangulation of points on polyhedral surfaces

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Abstract: It is well known that the complexity of the Delaunay triangulation of n points in R^d, i.e. the number of its simplices, can be $\Omega(n^{\lceil \frac{d}{2} \rceil})$. In particular, in R^3, the number of tetrahedra can be quadratic. Differently, if the points are uniformly distributed in a cube or a ball, the expected complexity of the Delaunay triangulation is only linear. The case of points distributed on a surface is of great practical importance in reverse engineering since most surface reconstruction algorithms first construct the Delaunay triangulation of a set of points measured on a surface.

In this paper, we bound the complexity of the Delaunay triangulation of points distributed on the boundary of a given polyhedron. Under a mild uniform sampling condition, we provide deterministic asymptotic bounds on the complexity of the 3D Delaunay triangulation of the points when the sampling density increases. More precisely, we show that the complexity is $O(n^{1.8})$ for general polyhedral surfaces and $O(n\sqrt{n})$ for convex polyhedral surfaces. Our proof uses a geometric result of independent interest that states that the medial axis of a surface is well approximated by a subset of the Voronoi vertices of the sample points. The proof extends easily to higher dimensions, leading to the first non trivial bounds for the problem when $d > 3$.

Key-words: Computational geometry, Delaunay triangulation, polyhedral surfaces, complexity, surface reconstruction

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Complexité de la triangulation de Delaunay de points distribués sur des surfaces polyédriques

Résumé : La complexité de la triangulation de Delaunay de n points de R^d, c'est-à-dire le nombre de ses faces, peut être $\Omega(n^{\lceil \frac{d}{2} \rceil})$. En particulier, dans R^3, le nombre de tétraèdres peut être quadratique. En revanche, si les points sont uniformément distribués dans un cube ou une boule, la complexité moyenne de la triangulation de Delaunay est linéaire. Le cas de points répartis sur une surface est d'un grand intérêt car la plupart des méthodes de reconstruction de surfaces utilisent la triangulation de Delaunay des points qui échantillonne la surface.

Dans cet article, nous bornons la complexité de la triangulation de Delaunay de points distribués sur le bord d'un polyèdre. Sous une hypothèse d'échantillonnage uniforme assez faible, nous majorons asymptotiquement la complexité de la triangulation de Delaunay tridimensionnelle quand la densité de l'échantillon augmente. Plus précisément, nous montrons que la complexité est $O(n^{1.8})$ pour des surfaces polyédriques générales et $O(n\sqrt{n})$ dans le cas convexe.

Notre preuve utilise un résultat géométrique intéressant en lui-même qui établit que le squelette d'une surface est bien approximé par un sous-ensemble des sommets de Voronoï du diagramme de Voronoï d'un échantillon de points sur la surface. La preuve s'étend sans difficulté aux dimensions supérieures, conduisant aux premières bornes non triviales pour le problème quand $d > 3$.

Mots-clés : Géométrie algorithmique, triangulation de Delaunay, surfaces polyédriques, complexité, reconstruction de surfaces

1 Introduction

It is well known that the complexity of the Delaunay triangulation of n points in $\mathbb{R}^d$, i.e. the number of its simplices, can be $\Omega(n^{\lceil \frac{d}{2} \rceil})$. In particular, in $\mathbb{R}^3$, the number of tetrahedra can be quadratic. Differently, if the points are uniformly distributed in a cube or a ball, the expected complexity of the Delaunay triangulation is only linear [8, 9].

The case of points distributed on a surface is of great practical importance in reverse engineering since most surface reconstruction algorithms first construct the Delaunay triangulation of a set of points measured on a surface, see e.g. [1, 4]. The time complexity of those methods therefore crucially depends on the complexity of the triangulation of points scattered over a surface in $\mathbb{R}^3$. Moreover, since output-sensitive algorithms are known for computing Delaunay triangulations [6], better bounds on the complexity of the Delaunay triangulation would immediately imply improved bounds on the time complexity of computing the Delaunay triangulation.

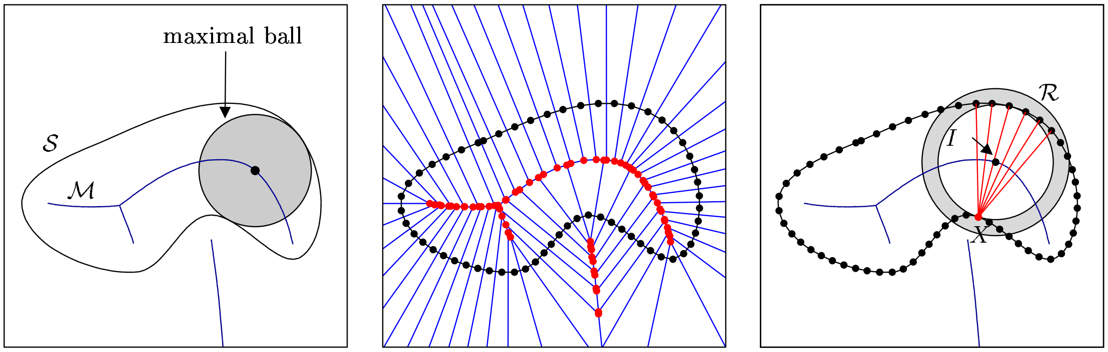
A first result has been recently obtained by Golin and Na [11]. They proved that the expected complexity of 3D Delaunay triangulations of random points on *convex* polytopes is $\Theta(n)$. The case of points on a cylinder has been considered by J. Erickson who proved that, even if the cylinder is well-sampled, the complexity of the Delaunay triangulation may be $\Omega(n\sqrt{n})$ [10]. Erickson's paper contains also lower bounds for contrived surfaces with a non bounded ratio between diameter and minimum local feature size, a case we exclude here.

In this paper, we consider the case of points distributed on the boundary of a given polyhedron. Under a mild uniform sampling condition, we provide deterministic asymptotic bounds on the complexity of the 3D Delaunay triangulation of the points when the sampling density increases. More precisely, we show that the complexity is $O(n^{1.8})$ for general polyhedral surfaces and $O(n\sqrt{n})$ for convex polyhedral surfaces. The intuition behind our result is the following. When a surface $\mathcal{S}$ is well-sampled, the circumcenters of the Delaunay simplices with a long edge are close to the medial axis $\mathcal{M}$ of $\mathcal{S}$ (see Figure 1b). It follows that the Delaunay neighbours of a point X that are sufficiently far away from X lie in a small region $\mathcal{R}$ between two spheres centered at a point I of $\mathcal{M}$ (see Figure 1c). In the case of a polyhedral surface, the intersection of $\mathcal{R}$ and $\mathcal{S}$ is contained in a bounded number of small disks and therefore X can only have a small number of Delaunay neighbours.

Our proof technique extends easily to higher dimensions, leading to the first non trivial bounds for the problem when $d > 3$.

2 Medial axis approximation

Our combinatorial bound on the complexity of the Delaunay triangulation of a light uniform sample $\mathcal{A}$ is based on a geometric result (Proposition 10 below) that states that the medial axis of the surface is well approximated by a subset of the Voronoi vertices of $\mathcal{A}$. Before



(a) (b) (c)

Figure 1: Overview of the proof in $\mathbb{R}^2$. (a) A curve $\mathcal{S}$ and its medial axis $\mathcal{M}$. (b) Voronoi graph of sample points. (c) The inner Delaunay neighbours of X that are far away from X lie in a small region $\mathcal{R}$.

stating and proving this result, we recall the definition of the medial axis of a surface and define uniform samples.

2.1 Definitions

The medial axis is a global description of objects, also called the skeleton. It has first been introduced by Blum in the field of image analysis as a tool for shape description. Since then, the medial axis has been intensively studied. In this section, we recall definitions and properties related to the medial axis. Other results on the medial axis can be found in [13, 7, 12, 14, 3].

The medial axis of a subset $\mathcal{F}$ of $\mathbb{R}^3$ can be defined using the concept of a maximal ball.

Definition 1 (Maximal ball) *Let $\mathcal{F} \subset \mathbb{R}^3$. A ball B is said to be* maximal *in $\mathcal{F}$ if and only if for any ball B', $B \subset B' \subset \mathcal{F} \Longrightarrow B = B'$.*

Definition 2 (Medial axis of an object) *Let $\mathcal{F} \subset \mathbb{R}^3$. The* medial axis *of $\mathcal{F}$ is the locus of the centers of the maximal balls of $\mathcal{F}$.*

This definition can be extended to surfaces as follows.

Definition 3 (Medial axis of a surface) *Let $\mathcal{S}$ be an embedded two-manifold. We call medial axis of $\mathcal{S}$ the medial axis of $\mathbb{R}^3 - \mathcal{S}$.*

At any point $X \in \mathcal{S}$, we associate the local feature size. The concept of local feature size has first been introduced in the context of surface reconstruction by Amenta and Bern [1].

Definition 4 (Local feature size) *The local feature size* $\text{lfs}(X)$ *at a point $X \in \mathcal{S}$ is the distance from X to the medial axis of $\mathcal{S}$.*

In the most general case, there can be an infinite number of maximal balls through a given point $X \in \mathcal{S}$. Let R be the radius of any maximal ball through X:

$$\text{lfs}(X) \leq R$$

However, if the normal to $\mathcal{S}$ at X is defined, there are exactly two maximal balls through X, on both sides of the tangent plane to $\mathcal{S}$ at X.

We distinguish two types of points on $\mathcal{S}$, singular and regular points:

Definition 5 (Singular and regular points) *A point $X \in \mathcal{S}$ is said to be* regular *iff 1. the normal to $\mathcal{S}$ at X is defined, 2. the two maximal balls through X touches $\mathcal{S}$, at least, at two distinct points X and $Y \neq X$. A point is said to be* singular *iff it is not regular.*

2.2 Uniform samples

Let $\mathcal{A} \in \mathcal{S}$ be a set of sample points of $\mathcal{S}$. We impose on $\mathcal{A}$ to be a uniform ε-sampling of $\mathcal{S}$. In addition, we enforce the sample not to become arbitrarily dense locally and introduce the notion of a light sample.

Definition 6 (Uniform ε-sample) *A set of points $\mathcal{A} \in \mathcal{S}$ is called a* uniform ε-sample *of $\mathcal{S}$ iff for every point $X \in \mathcal{S}$, the ball $B(X, \varepsilon)$ encloses at least one point of $\mathcal{A}$.*

Definition 7 (Light uniform ε-sample) *A uniform ε-sample is said to be* light *iff for every point $X \in \mathcal{S}$, the ball $B(X, r)$ encloses $O(\frac{r^2}{\varepsilon^2})$ points of $\mathcal{A}$.*

In particular, if the surface is bounded, the number of points n is bounded and $n = O\left(\frac{1}{\varepsilon^2}\right)$.

In this paper, we consider the surface to be fixed and provide asymptotic results when the sampling density increases, i.e. when ε tends to 0. In particular, we assume that quantities like the area or the diameter of $\mathcal{S}$ are fixed and do not depend on ε. Notice that our definition of a light uniform ε-sample does not impose any lower bound on the minimal distance between two sample points.

Amenta and Bern have introduced a different definition of an ε-sample [1]. The originality of their definition is to force the sample to fit locally the surface shape. According to their definition, point density is high where the surface has high curvature or where the object or its complement is thin. However, if the local feature size vanishes, an ε-sample, as defined in [1], will have an infinite number of points, which is not satisfactory for our purpose.

Erickson has introduced a notion of uniform sample that is related to our notion of light uniform sample but forbids points to be too close (which our definition allows) [10].

2.3 Medial axis approximation

The goal of this section is to prove that the circumcenters of the Delaunay tetrahedra with long edges converge towards the medial axis (Proposition 10). Our result improves on related results obtained by Amenta and Kolluri [2] and Boissonnat and Cazals [5].

Notations. In the rest of this section, X and A are sample points on the surface that are adjacent in the Delaunay triangulation. X is assumed to be a regular point of $\mathcal{S}$. I is the center of one of the two maximal balls through X. R is its radius. V designates a vertex of the Voronoi facet dual to the Delaunay edge $[XA]$.

Note : In this section as well as in the rest of the paper, we will provide first order approximations in ε and ignore higher order terms.

We start with a technical lemma that bounds $\|VI\|$, when the Delaunay edge $[XA]$ is long enough:

Lemma 8 *Let A, X, I and V be four points satisfying the following conditions:*
1. $\|VA\| = \|VX\|$,
2. $\|XI\| = R$,
3. $\|AI\| = R(1 + \rho)$, *for some small* $\rho \geq 0$,
4. $\angle(\overrightarrow{XI}, \overrightarrow{XV}) = \theta$, *for some small* $\theta \geq 0$,
5. $\|XA\| = 2l$ *with* $l^2 \gg \rho R^2$ *and* $l \gg R\theta$.

Ignoring second and higher order terms, we have

$$\|VI\| \leq \frac{\theta R^2}{l} + \frac{R^3 \rho}{2l^2}.$$

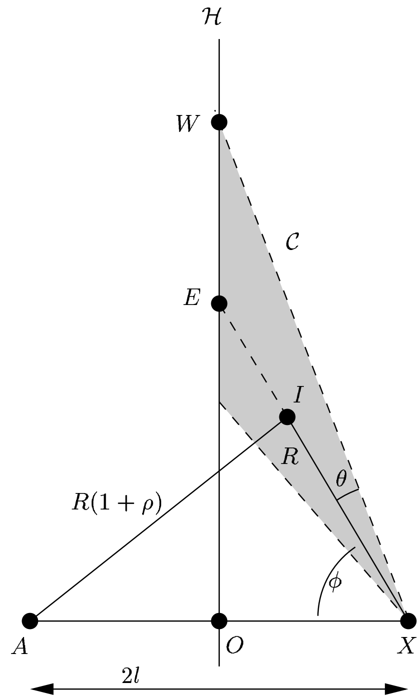


Figure 2: For the proof of Lemma 8.

Proof. Refer to Figure 2. Assume the three points A, X and I are given. Let $\mathcal{H}$ be the plane that bisects the points X and A. Let $\mathcal{C}$ be the right circular cone generated by the lines passing through X and forming an angle θ with the line (XI). Point V lies in plane $\mathcal{H}$ and on the cone $\mathcal{C}$. Therefore, V lies on the ellipsis $\mathcal{H} \cap \mathcal{C}$. Let W be the point on this ellipsis which is the farthest from point I. In addition, let O be the mid-point of segment $[AX]$ and E be the intersection point of line (XI) with $\mathcal{H}$. Let us bound $\|WI\|$. By triangle inequality:

$$\|WI\| \leq \|WE\| + \|EI\|$$

Let ϕ be the angle between the two vectors $\overrightarrow{XO}$ and $\overrightarrow{XI}$. Since the lengths of the three sides of triangle (AXI) are known, we have:

$$\cos\phi = \frac{-\rho(2+\rho)R^2 + 4l^2}{4lR} \approx \frac{l}{R} \gg \theta$$

Using the fact that $\cos\phi \gg \theta$, we can bound $\|WE\|$.

$$\begin{aligned}
\|WE\| &= l\,(\tan(\phi+\theta) - \tan\phi) \\
&= \frac{l\sin\theta}{\cos(\phi+\theta)\cos\phi} \\
&\approx \frac{l\theta}{\cos^2\phi} \\
&\approx \frac{\theta R^2}{l}
\end{aligned}$$

Let us now bound $\|EI\|$.

$$\begin{aligned}\|EI\| &= \frac{l}{\cos\phi} - R \\ &= \frac{4l^2R}{4l^2 - \rho(2+\rho)R^2} - R \\ &\approx \frac{R^3\rho}{2l^2}\end{aligned}$$

Finally, we get:

$$\|WI\| \leq \frac{\theta R^2}{l} + \frac{R^3\rho}{2l^2}$$

□

Let V denote a vertex of the Voronoi cell of X in the Voronoi diagram of $\mathcal{A}$. By slightly adapting a result of Amenta and Bern [1, Lemma 5], we can bound $\theta = \angle(\overrightarrow{XI}, \overrightarrow{XV})$.

Lemma 9 *Let X be a regular point of $\mathcal{S}$. Assume $\|VX\| \geq \varepsilon$ and $\mathrm{lfs}(X) \geq \varepsilon$. Then, the angle at X between the normal to $\mathcal{S}$ at X and the vector to V (oriented so that the angle is acute) is at most* $\arcsin\left(\frac{\varepsilon}{\|VX\|}\right) + \arcsin\left(\frac{\varepsilon}{\mathrm{lfs}(X)}\right)$.

The next proposition states that the circumcenters of the Delaunay tetrahedra with long edges converge towards the medial axis when ε tends to 0.

Proposition 10 (Medial axis approximation) *Let $\mathcal{A}$ be a uniform ε-sample of $\mathcal{S}$ and $X \in \mathcal{A}$ be a regular point of $\mathcal{S}$. Let $[XA]$ be a Delaunay edge and V a vertex in the Voronoi facet dual to the Delaunay edge $[XA]$. Let $B(I,R)$ be the one maximal ball through X such that $\overrightarrow{XV} \cdot \overrightarrow{XI} > 0$. Let Y be a point of $\partial B(I,R) \cap \mathcal{S}$ distinct from X. Assume that:*

1. $\varepsilon \ll \mathrm{lfs}(X)$,
2. $\frac{\|XY\|^2}{4R^2} \gg \frac{\varepsilon}{\mathrm{lfs}(X)}$,
3. $\frac{\|XA\|^2}{4R^2} \gg \frac{\varepsilon}{\mathrm{lfs}(X)}$.

Then:

$$\|VI\| \leq R\frac{\varepsilon}{\mathrm{lfs}(X)} \max\left(\frac{10R^2}{\|XY\|^2}, \left(1 + \frac{2R}{\|XA\|}\right)^2\right).$$

Proof. Since $\|VX\| \geq \frac{1}{2}\|XA\|$, $R \geq \mathrm{lfs}(X)$ and $\frac{\varepsilon}{\mathrm{lfs}(X)} \ll 1$, we have:

$$\frac{\varepsilon}{\|VX\|} \leq \frac{2\varepsilon}{\|XA\|} \ll \frac{\varepsilon}{R}\sqrt{\frac{\mathrm{lfs}(X)}{\varepsilon}} \leq \sqrt{\frac{\varepsilon}{\mathrm{lfs}(X)}} \ll 1 \tag{1}$$

Therefore, we can apply Lemma 9. The angle θ between the vectors $\overrightarrow{XI}$ and $\overrightarrow{XV}$ is at most :

$$\theta \leq \arcsin\left(\frac{\varepsilon}{\|VX\|}\right) + \arcsin\left(\frac{\varepsilon}{\mathrm{lfs}(X)}\right) \approx \frac{\varepsilon}{\|VX\|} + \frac{\varepsilon}{\mathrm{lfs}(X)} \tag{2}$$

Let Σ_v and Σ_p be two spheres passing through X. The first one, Σ_v, is centered at V and is therefore empty. The second one, Σ_p, has radius R and is centered on the half-line going from X to V. We denote by P the center of Σ_p.

Two cases must be considered. First, assume V is farther from X than P (see Figure 3 left). In this case, the ball bounded by Σ_p is contained in the ball bounded by Σ_v and therefore does not contain any point of $\mathcal{A}$ in its interior. Let A_y be one of the sample points in the neighbourhood of Y at distance at most ε from Y, and Σ_c the sphere tangent to Σ_p at X and passing through A_y. We denote by C the center of Σ_c. Since V lies between C and

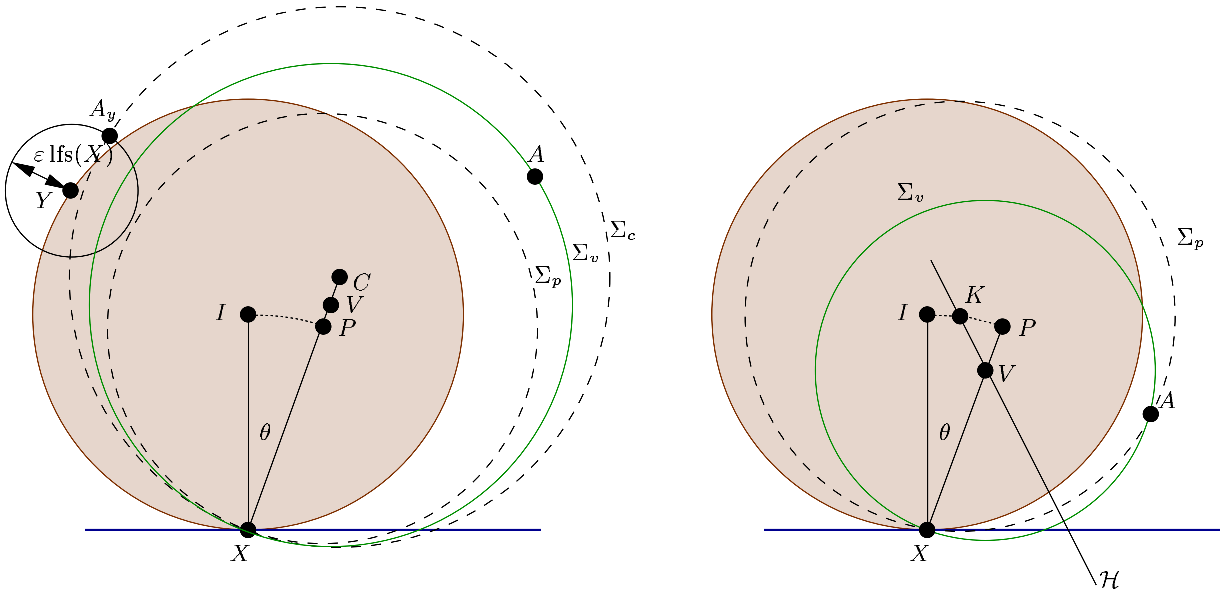


Figure 3: The two cases for the proof of Proposition 10

P, $\|VI\| \le \|CI\|$. We apply Lemma 8 with A_y, X, I and C. The five items of Lemma 8 are fulfilled. Indeed, since $\|VX\| \ge \|PX\| = R \ge \mathrm{lfs}(X)$, Equation (2) implies $\theta \le \frac{2\varepsilon}{\mathrm{lfs}(X)}$; as noticed above, $\rho \le \frac{\varepsilon}{R} \le \frac{\varepsilon}{\mathrm{lfs}(X)}$; and $2l = \|XA_y\| \ge \|XY\| - \varepsilon$. Let us first remark that $\|XY\| \gg \varepsilon$. Indeed, using again the inequality $R \ge \mathrm{lfs}(X)$, we have:

$$\frac{\|XY\|}{2} \gg R\sqrt{\frac{\varepsilon}{\mathrm{lfs}(X)}} \ge \sqrt{\mathrm{lfs}(X)\varepsilon} \gg \varepsilon \tag{3}$$

We now use our assumption $\frac{\|XY\|^2}{4R^2} \gg \frac{\varepsilon}{\mathrm{lfs}(X)}$ to prove that $l \gg R\theta$ and $l^2 \gg \rho R^2$. Indeed:

$$l \ge \frac{\|XY\|}{2} - \varepsilon \gg R\sqrt{\frac{\varepsilon}{\mathrm{lfs}(X)}} \gg R\frac{2\varepsilon}{\mathrm{lfs}(X)} \ge R\theta$$

$$\text{and} \quad l^2 \ge \left(\frac{\|XY\|}{2} - \varepsilon\right)^2 \gg R^2\frac{\varepsilon}{\mathrm{lfs}(X)} \ge \frac{\varepsilon}{R} \ge \rho$$

Since $\frac{R}{\|XY\|} \ge \frac{1}{2}$, Lemma 8 then implies:

$$\|VI\| \le \|CI\| \le \frac{R^2\theta}{l} + \frac{R^3\rho}{2l^2} \le \frac{4R^2}{\|XY\|}\frac{\varepsilon}{\mathrm{lfs}(X)}\left(1 + \frac{R}{2\|XY\|}\right) \le R\frac{\varepsilon}{\mathrm{lfs}(X)}\frac{10R^2}{\|XY\|^2} \tag{4}$$

Consider now the second case and assume that V is closer to X than P. Let $\mathcal{H}$ be the plane that bisects X and A. Since $\mathcal{H}$ contains V, P lies in the half-space limited by $\mathcal{H}$ that contains A. Since I lies in the other half-space, the circle arc IP (centered at X and of radius R) must intersect $\mathcal{H}$ at some point K. We now apply Lemma 8 to A, X, K and V. Notice that $\rho = 0$ and $\theta' = \angle(\overrightarrow{XK}, \overrightarrow{XV})$. Let $l = \frac{1}{2}\|XA\|$. Since $\rho = 0$, $l^2 \gg \rho R^2$. By Inequality 1, $\frac{\varepsilon}{\|VX\|} \ll \sqrt{\frac{\varepsilon}{\mathrm{lfs}(X)}}$ and thus:

$$\theta' \le \theta \le \frac{\varepsilon}{\|VX\|} + \frac{\varepsilon}{\mathrm{lfs}(X)} \ll \sqrt{\frac{\varepsilon}{\mathrm{lfs}(X)}} + \frac{\varepsilon}{\mathrm{lfs}(X)} \approx \sqrt{\frac{\varepsilon}{\mathrm{lfs}(X)}}$$

We now use our assumption $\frac{\|XA\|^2}{4R^2} \gg \frac{\varepsilon}{\mathrm{lfs}(X)}$ to prove that $l \gg R\theta'$:

$$l = \frac{1}{2}\|XA\| \gg R\sqrt{\frac{\varepsilon}{\mathrm{lfs}(X)}} \ge R\theta'$$

Lemma 8 then gives:

$$\|VK\| \leq \frac{R^2\theta'}{l}$$

Therefore:

$$\begin{aligned}
\|VI\| &\leq \|VK\| + \|KI\| \\
&\leq \frac{R^2\theta}{l} + R\theta \\
&\leq R\varepsilon\left(\frac{1}{\|VX\|} + \frac{1}{\mathrm{lfs}(X)}\right)\left(1 + \frac{2R}{\|XA\|}\right) \\
&\leq R\,\frac{\varepsilon}{\mathrm{lfs}(X)}\left(1 + \frac{2R}{\|XA\|}\right)^2
\end{aligned}$$

□

Remark that the proposition above makes no assumption on the contact point Y of the maximal ball through X. Y may or may not be regular. It does not matter either that the local feature size vanishes at Y.

We can also remark that the result depends on the ratio $\frac{\varepsilon}{\mathrm{lfs}(X)}$. In fact, if $r = \frac{\varepsilon}{\mathrm{lfs}(X)}$, we get an equivalent approximation theorem for r-samples, as defined by Amenta and Bern in [1].

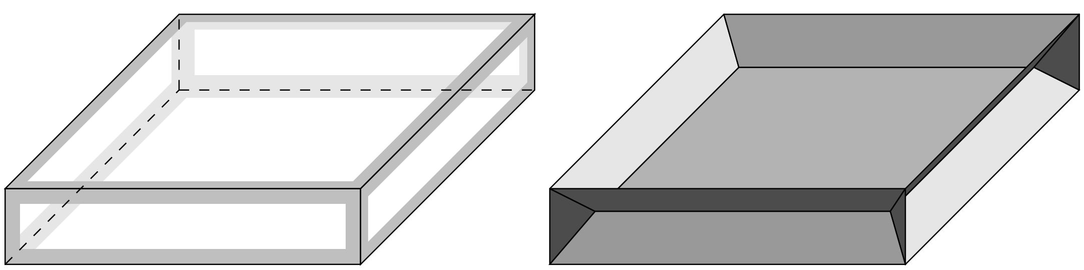
3 Polyhedral surfaces

3.1 Definition and properties

A polyhedral surface is the boundary of a bounded polyhedron. The medial axis of a polyhedral surface is composed of pieces of planes and quadrics (see Figure 4b). The singular points of a polyhedral surface are its edges and vertices. Polyhedral surfaces have two interesting properties:

Property 11 *Let $\mathcal{S}$ be a polyhedral surface. For any point $X \in \mathcal{S}$ and for any positive $r < \mathrm{lfs}(X)$, $\mathcal{S} \cap B(X, r)$ is a disk.*

Property 12 *Let $\mathcal{S}$ be a polyhedral surface and B a maximal ball centered on the medial axis of $\mathcal{S}$. If B touches $\mathcal{S}$ at two distinct points X and Y, $\|XY\| \geq 2\,\mathrm{lfs}(X)$.*



(a) (b)

Figure 4: A polyhedral surface and its l-singular zone on the left. Its medial axis on the right.

Definition 13 (Bounded polyhedral surfaces) *We say that a polyhedral surface is bounded if 1. it has a bounded number of faces, 2. the sum of the lengths of its edges is bounded.*

In the sequel, we restrict ourselves to *bounded polyhedral surfaces*. $\mathcal{S}$ will designate a bounded polyhedral surface and $\mathcal{A}$ a light uniform ε-sample of $\mathcal{S}$.

3.2 Counting Delaunay edges

In this section, we count the Delaunay edges incident to $X \in \mathcal{A}$. we enclose $\mathcal{S}$ in a sufficiently large bounding box $\mathcal{B}$. Therefore, the radius of any maximal ball remains bounded. We denote by $R_{\max}$ the radius of the greatest maximal ball. Moreover we add points on $\mathcal{B}$ so that the union of these additional points and the sample points on $\mathcal{S}$ constitute a light uniform ε-sample of $\mathcal{B} \cup \mathcal{S}$. Observe that the total number of points remains $O(n)$.

We consider two different types of zones on the surface (see Figure 4a), a l-singular zone surrounding singular points and a l-regular zone containing exclusively regular points.

Definition 14 (l-Regular and l-singular zones) *Let $l \geq 0$. We call* l-regular zone *of $\mathcal{S}$, the set of points $X \in \mathcal{S}$ such that* $\mathrm{lfs}(X) > l$. *We call* l-singular zone *of $\mathcal{S}$ the set of points that do not belong to the l-regular zone.*

The 0-singular zone (*resp.* the 0-regular zone) of a polyhedral surface consists of its singular (*resp.* regular) points. In this section, we search for which value of l the complexity in the two zones is counterbalanced.

In the l-regular zone, every point X has only two types of Delaunay neighbours: neighbours that are "close" to X and neighbours that are "close" to the two maximal balls through X:

Proposition 15 *Let $l^3 \gg 4R_{\max}^2\varepsilon$. Let X be a sample point in the l-regular zone of $\mathcal{S}$ and A, a Delaunay neighbour of X. Let $B(I_0, R_0)$ and $B(I_1, R_1)$ be the two maximal balls through X. Then :*

$$A \in B(X, \mathrm{lfs}(X)) \cup B\left(I_0, R_0 + 18\, R_0^3\, \frac{\varepsilon}{l^3}\right) \cup B\left(I_1, R_1 + 18\, R_1^3\, \frac{\varepsilon}{l^3}\right)$$

Proof. Let V be any vertex of the Voronoi facet dual to the Delaunay edge $[XA]$. Let I and R be the center and the radius of the maximal ball through X such that $\overrightarrow{XV} \cdot \overrightarrow{XI} > 0$. We assume $\|XA\| \geq \mathrm{lfs}(X)$ and prove that $A \in B\left(I, R + 8\, R^3\, \frac{\varepsilon}{l^3}\right)$.

In order to apply Proposition 10, we have to check that $\frac{\|XY\|^2}{4R^2} \gg \frac{\varepsilon}{\mathrm{lfs}(X)}$ and $\frac{\|XA\|^2}{4R^2} \gg \frac{\varepsilon}{\mathrm{lfs}(X)}$. But, by Proposition 12, $\|XY\| \geq 2\,\mathrm{lfs}(X)$ and:

$$\frac{\|XY\|^2}{4R^2} \geq \frac{\mathrm{lfs}(X)^2}{R^2} \geq \frac{l^2}{R_{\max}^2} \gg \frac{\varepsilon}{l} \geq \frac{\varepsilon}{\mathrm{lfs}(X)}$$

In a similar way:

$$\frac{\|XA\|^2}{4R^2} \geq \frac{\mathrm{lfs}(X)^2}{4R^2} \geq \frac{l^2}{4R_{\max}^2} \gg \frac{\varepsilon}{l} \geq \frac{\varepsilon}{\mathrm{lfs}(X)}$$

Therefore and using the inequality $1 \leq \frac{R}{\mathrm{lfs}(X)}$:

$$\|VI\| \leq R\, \frac{\varepsilon}{\mathrm{lfs}(X)} \max\left(\frac{10R^2}{\mathrm{lfs}(X)^2}, \left(1 + \frac{2R}{\mathrm{lfs}(X)}\right)^2\right) \leq 9\, R^3\, \frac{\varepsilon}{l^3}$$

An upper bound on $\|IA\|$ follows immediately since, by triangle inequality:

$$\begin{aligned} \|IA\| &\leq \|IV\| + \|VA\| = \|IV\| + \|VX\| \\ &\leq 2\|IV\| + \|IX\| \\ &\leq R\left(1 + 18\, R^2\, \frac{\varepsilon}{l^3}\right) \end{aligned}$$

□

Proposition 16 (Counting edges in the l-regular zone) *Let $l^3 \gg 4R_{\max}^2\varepsilon$. The number of Delaunay edges incident to a given sample point of the regular zone is $O(\frac{1}{\varepsilon\, l^3})$. The total number of edges incident to the l-regular zone is $O(\frac{1}{\varepsilon^3\, l^3})$.*

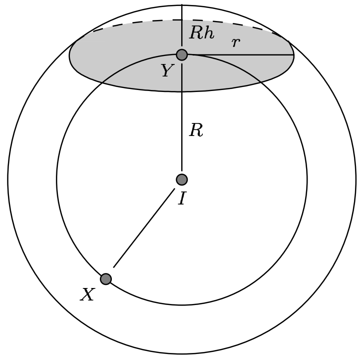


Figure 5: A plane tangent to the ball $B(I,R)$ at Y intersects $B(I,R+Rh)$ in a disk of radius $r = R\sqrt{2h+h^2} \approx R\sqrt{2h}$.

Proof. Let X be any point in the l-regular zone. Let $B(I_0,R_0)$ and $B(I_1,R_1)$ be the two maximal balls through X. The Delaunay neighbours of X belong to:

$$B(X,\mathrm{lfs}(X)) \cup B\left(I_0, R_0 + 18\, R_0^3\, \frac{\varepsilon}{l^3}\right) \cup B\left(I_1, R_1 + 18\, R_1^3\, \frac{\varepsilon}{l^3}\right).$$

First, we prove that the number of Delaunay neighbours in $B(X,\mathrm{lfs}(X))$ is $O(1)$. By Proposition 11, the intersection of the surface and $B(X,\mathrm{lfs}(X))$ is a disk $\mathcal{D}_x$. Let A be any Delaunay neighbour of X lying on this disk and let Σ_v be the empty sphere through X and A. Σ_v intersects $\mathcal{D}_x$ along a circle. Let C be the center of this circle and Σ_c be the sphere centered at C and passing through X and A. Σ_c is an empty sphere centered on $\mathcal{S}$. Because $\mathcal{A}$ is a uniform ε-sample, the radius of this sphere cannot be greater than ε. Therefore, $\|XA\| \leq 2\varepsilon$. By assumption, the number of sample points at distance 2ε from X is $O(1)$.

Secondly, we prove that the number of Delaunay neighbours in $B\left(I, R+18\,R^3\,\frac{\varepsilon}{l^3}\right)$ is $O(\frac{1}{\varepsilon\, l^3})$, where $B(I,R)$ designates any of the two maximal balls through X. For the sake of simplicity, let us call $h = 18\,R^2\,\frac{\varepsilon}{l^3}$ and $B(I,R+Rh)$ the enlarged maximal ball. Because the polyhedral surface has a bounded number of faces, the intersection of the enlarged maximal ball with $\mathcal{S}$ is included in a bounded number of disks. Each disk has radius at most $r \approx R\sqrt{2h} = 4R^2\sqrt{\frac{\varepsilon}{l^3}}$ (see Figure 5). Therefore, the number of sample points in the enlarged maximal ball is $O(\frac{r^2}{\varepsilon^2}) = O(\frac{1}{\varepsilon\, l^3})$. $\square$

Proposition 17 (Counting edges in the l-singular zone) *Let $l > 0$. The number of Delaunay edges joining two sample points in the l-singular zone is $O(\frac{l^2}{\varepsilon^4})$.*

Proof. Let p be the length of the 0-singular zone. The l-singular zone can be covered by $O(\frac{p}{l})$ spheres of radius $2l$. Therefore, the number of points in the l-singular zone is $O(\frac{p}{l} \times \frac{4l^2}{\varepsilon^2}) = O(\frac{l}{\varepsilon^2})$. The number of edges joining two points of the l-singular zone is therefore $O(\frac{l^2}{\varepsilon^4})$. $\square$

In order to counterbalance the number of edges in the two zones, we have to choose l such that $\frac{l^2}{\varepsilon^4} = \frac{1}{\varepsilon^3\, l^3}$, in other words $l = \sqrt[5]{\varepsilon}$. We sum up our results in the following theorem :

Theorem 18 *Let $\mathcal{A}$ be a light uniform ε-sample of a bounded polyhedral surface $\mathcal{S}$ of $\mathbb{R}^3$. The number of tetrahedra of the Delaunay triangulation of $\mathcal{A}$ is $O(n^{\frac{9}{5}}) = O(n^{1.8})$.*

4 Convex polyhedral surfaces

For convex polyhedral surfaces, the ratio $\frac{R}{\mathrm{lfs}(X)}$ is bounded from above by a certain constant C. Therefore, using the same scheme as before, one can prove that the complexity of convex polyhedral surfaces is $O(n\sqrt{n})$. Again, we consider two different zones on the convex polyhedral surface.

Proposition 19 *Let $l \gg \varepsilon$. Let X be a sample point in the l-regular zone of $\mathcal{S}$ and A, a Delaunay neighbour of X. Let $B(I_0, R_0)$ and $B(I_1, R_1)$ be the two maximal balls through X. Let $K = 2C(1+2C)^2$. Then :*

$$A \in B(X, \mathrm{lfs}(X)) \cup B(I_0, R_0 + K\varepsilon) \cup B(I_1, R_1 + K\varepsilon)$$

Proof. Let V be any vertex of the Voronoi facet dual to the Delaunay edge $[XA]$. Let I and R be the center and the radius of the maximal ball through X such that $\overrightarrow{XV} \cdot \overrightarrow{XI} > 0$ and assume $\|XA\| \geq \mathrm{lfs}(X)$. Let us prove that $A \in B(I, R + K\varepsilon)$.

Using $1 \gg \frac{\varepsilon}{l}$ and Property 12, we get :

$$\frac{\|XY\|^2}{4R^2} \geq \frac{\mathrm{lfs}(X)^2}{R^2} \geq \frac{1}{C^2} \gg \frac{\varepsilon}{l} \geq \frac{\varepsilon}{\mathrm{lfs}(X)}$$

In a similar way:

$$\frac{\|XA\|^2}{4R^2} \geq \frac{\mathrm{lfs}(X)^2}{4R^2} \geq \frac{1}{4C^2} \gg \frac{\varepsilon}{l} \geq \frac{\varepsilon}{\mathrm{lfs}(X)}$$

Therefore, by Proposition 10:

$$\|VI\| \leq R\,\frac{\varepsilon}{\mathrm{lfs}(X)}\left(1 + \frac{2R}{\mathrm{lfs}(X)}\right)^2 \leq \varepsilon\, C(1+2C)^2$$

and

$$\|IA\| \leq R + K\,\varepsilon$$

□

Proposition 20 (Counting edges in the l-regular zone) *Let $l \gg \varepsilon$. Let $\mathcal{A}$ be a light uniform ε-sample. The number of Delaunay edges incident to a given sample point of the l-regular zone is $O(\sqrt{n})$. The total number of edges incident to the l-regular zone is $O(n\sqrt{n})$.*

Proof. We have already proven that the number of Delaunay neighbours in $B(X, \mathrm{lfs}(X))$ is $O(1)$. Let us prove that the number of points in the enlarged maximal ball $B(I, R + K\,\varepsilon)$ is $O(\sqrt{n})$. The intersection of the enlarged maximal ball with $\mathcal{S}$ is included in a bounded number of disks. Each disk has radius at most $r = \sqrt{2RK\varepsilon}$ (see Figure 5). Therefore, the number of sample points in the enlarged maximal ball is $O(\frac{r^2}{\varepsilon^2}) = O(\frac{1}{\varepsilon}) = O(\sqrt{n})$. □

The counting of edges in the l-singular zone is unchanged and is given by Proposition 17. In order to find $O(n\sqrt{n}) = O(\frac{1}{\varepsilon^3})$ edges in the singular zone, we have to choose $\frac{l^2}{\varepsilon^4} = \frac{1}{\varepsilon^3}$, in other words $l = \sqrt{\varepsilon}$.

Theorem 21 *Let $\mathcal{A}$ be a light uniform ε-sample of a bounded convex polyhedral surface $\mathcal{S}$ of $\mathbb{R}^3$. The number of tetrahedra of the Delaunay triangulation of $\mathcal{A}$ is $O(n\sqrt{n})$.*

5 Conclusion

Except Propositions 16, 17 and Theorem 18, all our results can easily be extended to piecewise-linear $(d-1)$-manifolds of $\mathbb{R}^d$ for $d > 3$. In particular, for convex polytopes, the $O(n\sqrt{n})$ bound holds in any dimension. Propositions 16 and 17 can easily be adapted so as to work in $\mathbb{R}^d$. We simply need to count points in a $(d-1)$-ball. The volume of a $(d-1)$-ball with radius r is $O(r^{d-1})$ and therefore a $(d-1)$-ball contains $O(\frac{r^{d-1}}{\varepsilon^{d-1}})$ sample points. The complexity of the Delaunay triangulation of a light uniform sample on a general piecewise linear manifold is $O(n^{\frac{6(2d-3)}{7d-11}})$.

We have given a sub-quadratic bound for polyhedral surfaces. The case of smooth surfaces seems to be harder, due to the presence of slivers, i.e. flat tetrahedra whose circumcenters can be arbitrarily far from the medial axis. Observe that Proposition 10 indicates that these tetrahedra are small.

An obvious open question is to extend our results to smooth surfaces. We also suspect that our general bound is not tight and should be improved. In the convex case, the gap between our deterministic bound and the probabilistic result of Golin and Na asks for further work.

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